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Creation of kink and antikink pairs forced by radiation

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Abstract

The interaction between kink and radiation in nonlinear one-dimensional real scalar field is investigated. The process of discrete vibrational mode excitation in ϕ^4 model is considered. The role of these oscillations in the creation of kink and antikink is discussed. Numerical results are presented as well as some attempts of analytical explanations. An intriguing fractal structure in parameter space dividing regions with and without creation is also presented.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Topological defects are usually almost compact (with only some exponential tails), static solutions with finite energy of partial differential equations whose properties are defined by field values at infinity. The simplest examples of topological defects are one-dimensional kinks (ϕ^4 or sine-Gordon equation). Less trivial examples (vortices, domain walls, monopoles, etc) manifest themselves in various branches of physics [1, 2], such as particle physics [3], condensed matter physics, cosmology [4, 5] and much more.

Because of their stability and localized energy density some topological defects have similar properties as particles (e.g., kinks in 1+1 dimensions or monopoles in 3+1 dimensions). They can be created or annihilated as ordinary particles. They can also interact with each other, with radiation or external force. Reference [6] brings a very nice example of topological defects accelerating under a constant external force.

However, topological defects reveal some properties which make them very different from particles. In our previous paper [7], we have presented an unexpected behaviour when the kink in ϕ^4 was exposed to monochromatic radiation of scalar field coming from one direction. The nonlinear character of this interaction resulted in 'negative radiation pressure'. The kink, instead of being pushed by radiation, was in fact being pulled towards the source of this radiation. Similar feature reveals the second most often used example, the sine-Gordon soliton.

Another peculiarity of topological defects occurred in a two kink collision process in ϕ^4 theory [8] during which the kinks could be scattered back or annihilated. Let us stress out that similar process in sine-Gordon equation, because of its integrability, is less interesting. If the S-G solitons have enough energy they go through each other with nothing but a phase shift. When the energy is below some threshold the solution describes a bound state (breather) which is a perfectly periodic solution. ϕ^4 has much more interesting structure of solutions. During collision the kinks loose some of their initial energy on behalf of radiation and when the loss is small the kinks reflect from each other. When the loss is large they glue to each other forming a bound state (so-called oscillon [9, 10]). The bound state also radiates and finally vanishes. An extraordinary thing about this process is that there exists no real threshold. The regions in phase space for one or the other scenario mix and form a fractal similar to a Cantor's set.

This is not an isolated example. Goodman *et al* [11] bring a different example when a sine-Gordon soliton interacts with an oscillating impurity mode. They also observed a fractal structure.

We found yet another example of fractal structure but in a quite different process. The ϕ^4 soliton possesses an internal degree of freedom—an oscillational mode (in [8] authors claim it is responsible for a peculiar behaviour of kinks during collision). When that mode is excited, it oscillates with a certain frequency. In linear approximation, the mode oscillates infinitely long with constant amplitude. When one includes nonlinear correction, one finds the mode couples to scattering spectrum and radiates reducing its amplitude and finally vanishes (similar process but in much more complicated case of 't Hooft–Polyakov monopole was discussed in [12]). But when the initial amplitude is large enough so that its energy is just a little above a mass of two kinks, a kink and antikink are produced.

In the present work, we investigate an opposite process. We light radiation far away from the kink from both directions so that the kink would stand still. The radiation excites the oscillational mode and if the radiation amplitude and frequency is suitable the oscillational mode breaks up to kink and antikink. Of course, if radiation is large enough the kinks can be created even from vacuum, but because the oscillational mode gathers the energy our process seems to be much more efficient. One surprising result we have found is that in a plane radiation amplitude versus radiation frequency the border separating solutions with and without creation is also a fractal. We claim that the nonlinear coupling between the vibrational mode and radiation is responsible for the structure.

Our paper is organized as follows. In the following section, we give an introduction to the model discussed. We recall a spectral structure of linearization around the kink. The next section is devoted to excitation of the oscillational mode. We present our numerical results and attempts of theoretical explanation. We present a simple model reproducing qualitative results of the full model. Although on a precise level there are some discrepancies, the general structure of solutions remains similar. The next section is a brief description of the numerical method used. The last section is as usual conclusions and discussion.

We have also encountered a very interesting question of how one can simplify and reduce a system from infinite number of degrees of freedom to only few.

2. The model

Let us consider one-dimensional real scalar field obeying the equation in the form

$$\ddot{\phi} - \phi'' + U'(\phi) = 0, \tag{1}$$

where $U(\phi)$ is a potential with at least two equal minima (vacua) ϕ_v (without loss of generality we can assume that $U(\phi_v) = 0$) so that in these theories there exist static soliton solutions ϕ_s described by the equation

$$\phi_s(x) : x - x_o = \pm \int_{\phi_{v_1}}^{\phi_{v_2}} \frac{\mathrm{d}\phi}{\sqrt{2U(\phi)}}. \tag{2}$$

In this section, we will discuss the two most known models, i.e., ϕ^4 $\left(U(\phi) = \frac{1}{2}(\phi^2 - 1)^2\right)$ and sine-Gordon $(U(\phi) = 1 - \cos(\phi))$ equations. In these theories, the static soliton solutions have the form

$$\begin{cases} \phi_s(x) = \pm\tanh(x - x_o) & \text{for } \phi^4 \\ \phi_s(x) = \pm\arctan(\mathrm{e}^{-(x-x_0)}) & \text{for S-G.} \end{cases} \tag{3}$$

Because of the translational invariance we can substitute $x_0 = 0$.
Let us add some small perturbation to the static kink solution $(\phi = \phi_s + \xi)$. If the potential can be expanded in a Taylor' series (one can do so for many systems with an exception of certain compactons [13]):

$$U'(\phi) = U'(\phi_s + \xi) = U'(\phi_s) + U''(\phi_s)\xi + N(\xi, \phi_s) \tag{4}$$

where $N(\xi, \phi_s)$ is a nonlinear part in ξ $(N_{\phi^4} = 6\phi_s\xi^2 + 2\xi^3)$. We can write the equation for ξ as

$$\ddot{\xi} - \xi'' + V(x)\xi + N(\xi, x) \equiv \ddot{\xi} + \hat{L}\xi + N(\xi, x) = 0, \tag{5}$$

where

$$V(x) = U''(\phi_s(x)) = \begin{cases} 1 - \dfrac{2}{\cosh^2 x} & \text{for S-G} \\ 4 - \dfrac{6}{\cosh^2 x} & \text{for } \phi^4. \end{cases} \tag{6}$$

We substitute $\xi(t, x) = \exp(\mathrm{i}\omega t)\eta_k(x)$ and find the eigenvalues and the eigenfunctions of the operator $\hat{L} = -\mathrm{d}^2/\mathrm{d}x^2 + V(x)$. We can divide the spectra into the following three groups:

(i) translational zero modes: $\eta_t = \phi_s'$: $\phi_s(x + \delta x) \approx \phi_s(x) + \delta x\phi_s'(x)$, $\omega_t = 0$,

(ii) discrete oscillational modes (there is none for S-G and only one for ϕ^4 but in general there can be any finite number):

$$\eta_d^{(\phi^4)}(x) = \frac{\sinh x}{\cosh^2 x}, \qquad \omega_d = \sqrt{3},$$

(iii) scattering modes:

$$\eta_k(x) = \begin{cases} \mathrm{e}^{\mathrm{i}kx}(\mathrm{i}k - \tanh x) & \text{for S-G} \\ \mathrm{e}^{\mathrm{i}kx}(3\tanh^2 x - 1 - k^2 - 3\mathrm{i}k\tanh x) & \text{for } \phi^4, \end{cases}$$

where $k^2 = \omega^2 - m^2$, $m = 2$ for ϕ^4 and $m = 1$ for S-G.

It is very significant that scattering modes in these models have no reflection part (this is why the negative radiation pressure is possible [7]). This is not a general feature. In fact it is quite rare, but spectra for $V(x)$ given in a form

$$V(x) = -\frac{N(N+1)}{\cosh^2 x}$$

are reflectionless for all integer N. There are exactly N bounded modes. One of them is a translational mode and the rest of them are oscillational [14].

The existence of the oscillational mode in the ϕ^4 theory is responsible for its nonintegrability. During collisions, this mode is excited and takes some of the initial energy. Then the energy, due to the nonlinearities, is transferred to radiation modes.

Numerical results show that the radiation coming from infinity and hitting the kink can cause the creation of the pair kink and antikink. This process is much easier for ϕ^4. Even small amplitudes can force the process. In sine-Gordon's case, the process occurs for amplitudes compared to the amplitudes when the creation can occur even far away from kink in pure radiation. The explanation is quite simple. In ϕ^4, there exists an oscillation mode which can be excited by the radiation. Manton [15] investigated the creation of kink and antikink caused only by a discrete mode. When the energy contained in this mode was a little above than the energy of two kinks, a pair of kink and antikink was created and two kinks were radiated to infinity in opposite directions and antikink remained. When the energy was not large enough, the discrete mode oscillated with a decreasing amplitude and the energy was radiated to infinity because of the coupling to the scattering modes. In the following section, we will discuss an opposite process. Radiation coming from infinity will couple to the oscillating mode.

3. Excitation of the oscillating mode and the creation process in ϕ^4 model

3.1. Numerical results

We used two different sets of initial and boundary conditions to determine the coupling between radiation and the oscillating mode. In both cases we considered only antisymmetric problems, we evaluated our system only on one-half of the x-axis and posed the boundary condition $\phi(t,0) = 0$. We could do so because the symmetrical radiation does not couple with an oscillational mode and therefore gives no contribution. The symmetrical radiation couples however with a translational mode and the kink as a whole would start moving and would be a difficult object to study.

In the first case, we had initial conditions $\phi(0,x) = \phi_s(x)$. The boundary conditions were $\dot{\phi}(0,x) = 0$ and $\phi(t,L) = 1 + \frac{1}{2}A\sin\omega t$ (where L was large in comparison to the size of the kink, usually $L = 200$). We use $\frac{1}{2}$ because when the waves come from both sides of the kink, they add and produce a standing wave with twice as big amplitude. Because of the condition $\phi(t,0) = 0$ and the symmetry of the equation $\phi \to -\phi, x \to -x$, we actually studied the system with boundary conditions $\phi(t,\pm L) = 1 \pm \frac{1}{2}A\sin\omega t$. These conditions describe the kink in radiation coming from both sides from very far away. In order to measure the excitation of the discrete mode, we calculated the value (a projection on a discrete mode)

$$A_d(t) = \frac{\int_{-\infty}^{\infty} \mathrm{d}x\, \eta_d(x)[\phi(t,x) - \phi_s(x)]}{\int_{-\infty}^{\infty} \mathrm{d}x\, \eta_d^2(x)}. \tag{7}$$

Because the radiation is orthogonal to η_d there is no contribution to $A_d(t)$ from any other modes (at least in linear approximation). We measured A_d for different ω and A.

In figures 1–3, we have sketched three examples of A_d versus time for $\omega = 3.5$ and for A equals 0.2, 0.3 and 0.5.

For all figures, we can see that the reaction of the oscillating mode is retarded by about 240. Because the radiation is 'switched on' at $x = L$ it needs time $t_0 = \partial k/\partial\omega L = \omega L/k \approx 243$ to get to the kink.

If the amplitude is small (figure 1), A_d oscillates with eigenfrequency ω_d. These oscillations are modulated by another frequency. As we will show in the following section this modulating frequency is $\omega_m = \omega - 2\omega_d$ (figure 4). In our example, $\omega_m = 0.0359$ and the period is 175 so there is a very good agreement.

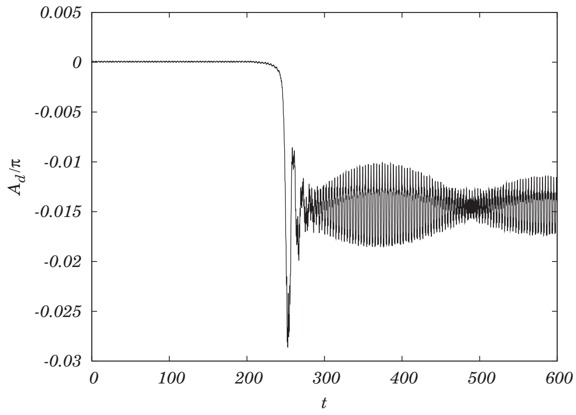

Figure 1. The excitation of the discrete mode for $A = 0.2$ and $\omega = 3.5$.

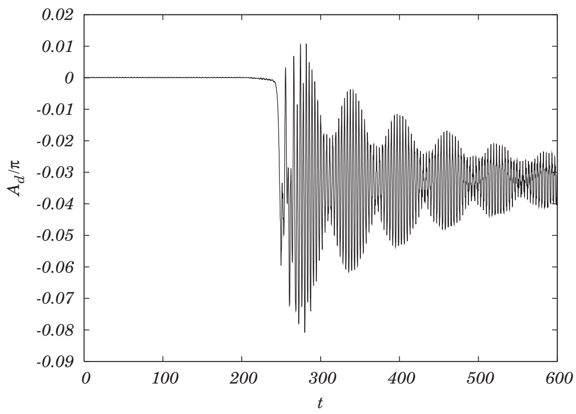

Figure 2. The excitation of the discrete mode for $A = 0.3$ and $\omega = 3.5$.

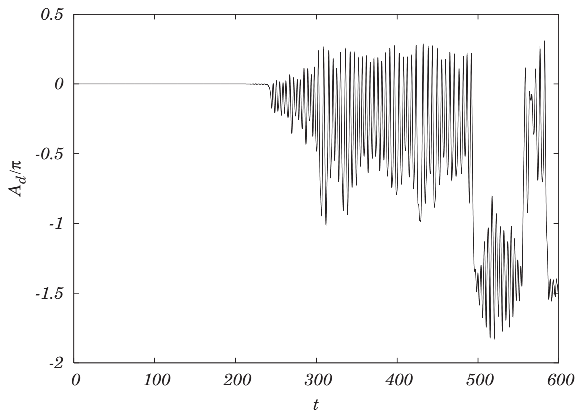

Figure 3. The excitation of the discrete mode for $A = 0.5$ and $\omega = 3.5$. One can see the creation of kink–antikink pairs for $t \approx 500$, $t \approx 560$ and $t \approx 590$.

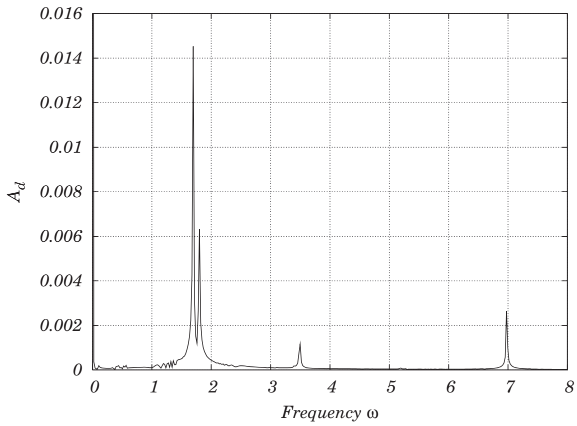

Figure 4. Fourier transform of A_d for $\omega = 3.5$ and $A = 0.2$. One can see four separated peaks for $\omega = \sqrt{3} \approx 1.73$, 3.5, 7 and $3.5 - \sqrt{3} \approx 1.76$.

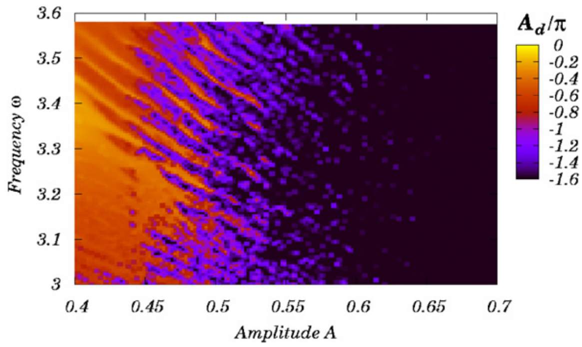

Figure 5. Minima of A_d versus frequency ω and amplitude A of radiation coming from $L = 200$. Dark spots $A_d < -\frac{3\pi}{2}$ represent creation process.

If the amplitude is larger the modulations are with higher frequency and the oscillations decay. This decay can be explained because as Manton showed in [15] the oscillating mode couples to the scattering modes and radiates its energy (see also [16, 17]).

The next figure (figure 3) is the most interesting. First, we can see some chaotic oscillations and then around $t = 500$ A_d suddenly goes to about -1.5π and then oscillates around that level, and then goes back to 0 (around $t = 560$). If the amplitude of A_d is large enough, a pair of kink and antikink can be created and radiated in both directions, but the place of a kink in the middle is taken by an antikink. When we substitute $\phi(x, t) = -\phi_s(x)$ into (7) we obtain that $A_d = -\frac{3\pi}{2}$. Figure 5 presents the minima of $A_d(t)$ as a dependence upon frequency ω and amplitude A of incoming radiation. When A_d is less than $-\frac{3\pi}{2}$, a creation occurred (dark spots). One can see that the border between creation and small oscillations is very complicated, probably fractal. The resolution is not good enough to prove this proposition but figure 6 may justify our hypothesis. We present the dependence $\ln n_b$ versus $\ln 1/l$ where n_b is a number of boxes containing the boundary and l is the relative size of the boxes. The slope of the fitted line should give a fractal dimension d_c. In this case, $d_c = 1.69 \pm 0.02$ and that would justify that the border is fractal. Of course, proving numerically that something is a fractal is very difficult and in our case is almost impossible.

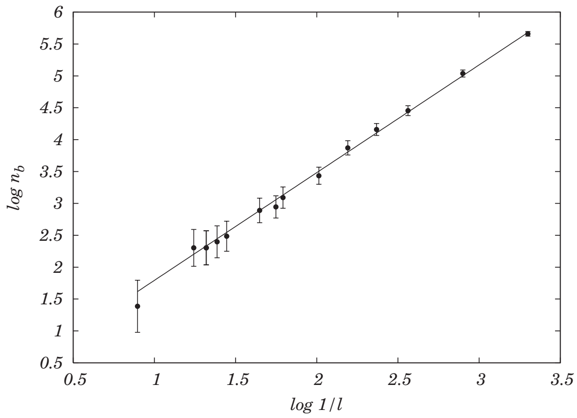

Figure 6. Dependence $\ln n_b$ upon $\ln 1/l$. The slope of the fitted line is 1.69 ± 0.02.

Although the case described above has a nice physical interpretation, because we switch on the light in one particular moment, it is very difficult to explain it in details using a simplified method presented in the next section. The reason is that the simplified equations for A_d are derived for monochromatic wave coming from both sides. When we consider the boundary conditions described in the beginning of the present section, we find that the radiation has a form of a very wide wavelet. Despite nonlinearities in the equation, the wavelet is always a superposition of many monochromatic waves. Although we can assume that after a long time we have almost a single frequency standing wave there is still a problem of initial conditions for $A_d(t_0)$. We found that the evolution of $A_d(t)$ highly depends on the initial conditions. We cannot tell whether A_d would oscillate or jump to $-\frac{3\pi}{2}$ (and force the creation of soliton pairs) if the initial conditions are not defined with enough accuracy.

It is much easier to consider the Cauchy problem with conditions $\xi(t = 0, x) = Ar_k(t = 0, x)$, where $r_k(t, x)$ is a real combination of scattering modes $\eta_{\pm k}$ in a form of a standing wave $r_k(t, x) = h_k(x) \cos \omega t$, where

$$h_k(x) = \frac{(3\tanh^2 x - 1 - k^2)\sin kx - 3k \tanh x \cos kx}{\sqrt{(k^2 + 4)(k^2 + 1)}}. \tag{8}$$

The denominator was introduced just for convenience so that the amplitude of the wave far away from the kink would be 1. Basically, the structure of the solutions looks the same but for different parameters A and the oscillations begin for $t = 0$. We have plotted an analogical figure to figure 5 (figure 7). We can see that the border between the two types of solutions (with a creation (dark spots) and without) also seems to be fractal (figure 8).

If we make a similar figure using the longer evolution, the border would be shifted in the direction of smaller A for obvious reasons. We tried to evolve our system as long as the border does not change significantly with longer time. Unfortunately, this procedure was very difficult to apply and we cannot be certain weather the border presented in figure 7 is a true limit, but we think it is a quite good approximation nevertheless.

3.2. Theory

Let us now try to explain the fractal structure using more analytical approach. In the present section, we will construct a simplified equation for the evolution of A_d. Our goal is not to give a precise nor formal analytical calculation but rather to point out the important features

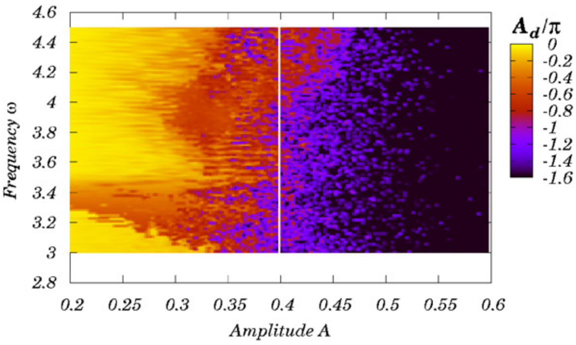

Figure 7. Minima of A_d versus frequency ω and amplitude A of radiation in the case of a standing wave.

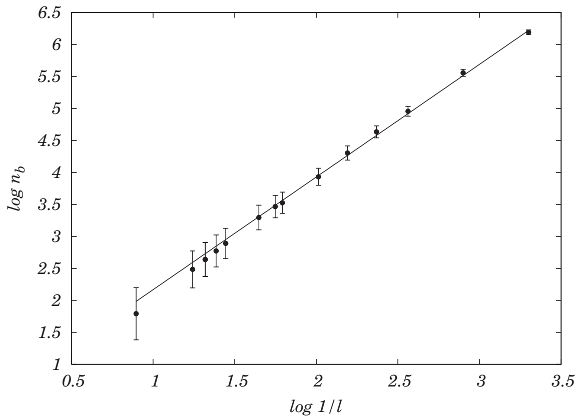

Figure 8. Dependence $\ln n_b$ upon $\ln 1/l$ for the second case. The slope of the fitted line is 1.76 ± 0.02.

of the system which are responsible for the formation of the fractal structure described in the previous section. Let us consider a simplified field equation for ξ in the ϕ^4 model (5):

$$\ddot{\xi} + \hat{L}\xi + 6\phi_s\xi^2 = 0. \tag{9}$$

We have neglected only the cubic term.

Let us decompose ξ into the sum of given radiation, excitation of the discrete mode and some orthogonal field η (for completeness):

$$\xi(t,x) = Ar_k(t,x) + A_d(t)\eta_d(x) + \eta(t,x). \tag{10}$$

If amplitude of the radiation A is small and there are no free oscillations of A_d nor η, it is consistent to assume that A_d and η are both of order $O(A^2)$. After substitution (10) into (9), we obtain (in the second order in A)

$$\begin{aligned}\left(\ddot{A}_d + \omega_d^2 A_d\right)\eta_d + \ddot{\eta} + \hat{L}\eta + 6\phi_s\big(\tfrac{1}{2}A^2h_k^2(1+\cos 2\omega t) + A_d^2\eta_d^2 + \eta^2 \\ + 2A_d\eta_d A h_k\cos\omega t + 2A_d\eta\eta_d + 2Ah_k\eta\cos\omega t\big) = 0. \end{aligned}\tag{11}$$

The first correction to A_d originated from the interaction between A_d and η would be of order $O(A^4)$ and we will neglect this term. Since η is orthogonal to η_d, we can multiply both sides of this equation by η_d and integrate (similar procedure was applied in [6]). The projected equation yields

$$\ddot{A}_d + \omega_d^2 A_d + A^2\alpha(k)(1 + \cos 2\omega t) + \beta(k) A A_d \cos\omega t + \gamma A_d^2 = 0, \tag{12}$$

where

$$\alpha(k) = 3\frac{\int_{-\infty}^{\infty} \mathrm{d}x\, \phi_s(x) h_k^2(x)\eta_d(x)}{\int_{-\infty}^{\infty} \mathrm{d}x\, \eta_d^2(x)}, \tag{13}$$

$$\beta(k) = 12\frac{\int_{-\infty}^{\infty} \mathrm{d}x\, \phi_s(x) h_k(x)\eta_d^2(x)}{\int_{-\infty}^{\infty} \mathrm{d}x\, \eta_d^2(x)}, \tag{14}$$

$$\gamma = 6\frac{\int_{-\infty}^{\infty} \mathrm{d}x\, \phi_s(x)\eta_d^3(x)}{\int_{-\infty}^{\infty} \mathrm{d}x\, \eta_d^2(x)} = \frac{9\pi}{16}. \tag{15}$$

The first two integrals can also be calculated analytically:

$$\alpha(k) = \frac{9\pi}{64N^2}(8k^4 + 34k^2 + 17)\left(1 - \frac{1}{\cosh k\pi}\right), \tag{16}$$

$$\beta(k) = 3\pi k^2 \frac{k^4 + 2k^2 - 8}{8N \sinh\frac{k\pi}{2}}, \tag{17}$$

where $N = \sqrt{(k^2+4)(k^2+1)}$.

Equation (12) is very similar to Mathieu's equation but with extra driving force and nonlinear term. The equation describes the evolution of A_d but during its derivation we used many simplifications. The oscillating mode couples to the scattering modes and is a source of radiation itself [7, 15]. This radiation carries away the energy from the discrete mode causing the mode to decay. Although we have predicted the loss of energy due to the radiation in [7] but the present case is more complicated. The nonlinear part in equation (11) contains sources in the form of $r_k A_d \eta_d$ and we do not know how much energy is taken from the oscillational mode and how much from original radiation. Finally, we do not know how this loss would decrease amplitudes of the nonharmonic oscillations of A_d. Moreover, the damping term plays less important role in the formation of the fractal structure. This is why we did not add the damping term.

Although equation (12) is nonlinear and in order to find the solutions we need to use numerical methods (which will be presented further in this section), one can find some features of the solution using the perturbation approach. Let us find a solution of the equation in a form (we do not wish to have homogeneous oscillations of the discrete mode at this point so that A_d could be of order $O(A^2)$)

$$A_d = A_d^{(2)} A^2 + A_d^{(3)} A^3 + A_d^{(4)} A^4 + \cdots$$

with a condition that A is very small. The substitution into (12) gives (in order $O(A^2)$)

$$\ddot{A}_d^{(2)} + \omega_d^2 A_d^{(2)} + \alpha(1 + \cos 2\omega t) = 0. \tag{18}$$

This equation is a simple forced harmonic oscillator equation and we can solve it very easily:

$$A_d^{(2)}(t) = -\frac{\alpha}{\omega_d^2} + \frac{\alpha}{4\omega^2 - \omega_d^2}\cos 2\omega t + B^{(2)}\cos(\omega_d t + \delta), \tag{19}$$

where $B^{(2)}$ is a homogeneous oscillation amplitude (now we can add this term and the order of the solution agrees). The first term $-\alpha\big/\omega_d^2$ describes the level of oscillation which is below 0 (figures 1 and 2).

Now we can also apply initial conditions such as $A_d(0) = 0$, $\dot{A}_d(0) = 0$ to calculate $B^{(2)}$ and δ. Instead of that let us assume, for simplicity, $\delta = 0$. The next order equation has the form

$$\ddot{A}_d^{(3)} + \omega_d^2 A_d^{(3)} + \beta A_d^{(2)} \cos \omega t = 0, \tag{20}$$

and solve it as well. The inhomogeneous term is equal to

$$\beta A_d^{(2)} \cos \omega t = \alpha\beta \left(-\frac{\cos \omega t}{\omega_d^2} + \frac{\cos 3\omega t + \cos \omega t}{2(4\omega^2 - \omega_d^2)} \right) + \frac{1}{2} B^{(2)} \beta (\cos(\omega_d + \omega)t + \cos(\omega - \omega_d)t). \tag{21}$$

The last term oscillates with a frequency $\omega - \omega_d$. This frequency is visible in the power spectrum in figure 4. In our example, the frequency $\omega - \omega_d \approx 1.76$ is not very different from ω_d which is an eigenfrequency of the internal mode. It is not a surprise that this term gives particularly large contribution to the evolution of A_d since it is very close to the resonance frequency. In figure 4, the amplitude at this frequency is the second dominating frequency. When two oscillations with similar frequencies add, one can observe modulation with a frequency which is a difference of these frequencies (in our case it is $\omega_m = \omega - 2\omega_d \approx 0.0359$). Even small $B^{(2)}$ can give a large contribution in the next order since the considered term can be very close to the resonance. Our perturbation scheme breaks down. If $\omega = 2\omega_d$, we could apply so-called *cancellation of the resonance term* method [18]. This method leads to the change in oscillation frequency. The most interesting question however is how big the amplitude A_d can be. We neglected some terms which are important to solve this problem. First of all we have presented only the solution in $O(A^3)$ order and the perturbation scheme cannot give the answer for large amplitude. We also neglected the cubic term and of course the field η which is radiation escaping from the discrete mode. We do not know how small A should be around the resonance frequency so that the oscillating mode amplitude would remain small during the whole evolution.

Let us now focus on the equation for $\omega = 2\omega_d$. We could choose any frequency but for simplicity we choose this one (this is also the resonance case). Although we can find the precise values of the coefficients $\alpha(k)$ and $\beta(k)$ for $k = 2\sqrt{2}$ for this particular model, we think it is better to examine the whole equation for any set of coefficients. For different models (different $U(\phi)$), there should be different values (α, β and γ) but the structure of the equation remains the same. First, we simplify the equation in order to get rid of too many parameters. We introduce

$$\begin{cases} w = \dfrac{\omega_d^2}{\gamma} A_d \\ \tau = \omega_d t \\ g_1 = \dfrac{\beta}{\omega_d^2} A \\ g_2 = \dfrac{\alpha\gamma}{\omega_d^4} A^2 \end{cases} \tag{22}$$

and our equation takes the form

$$\ddot{w} + w + w^2 + g_1 w \cos 2\tau + g_2(1 + \cos 4\tau) = 0, \tag{23}$$

where $\dot{}$ denotes $\mathrm{d}/\mathrm{d}\tau$. Numerical simulations of the above equation suggest that the solutions can be divided into three groups (due to the initial conditions and values of g_1 and g_2). In the first group, the solutions oscillate around some level w_0 with frequencies $1, 2, \ldots$ and are modulated with a very small frequency. The solutions have sometimes quite large amplitude,

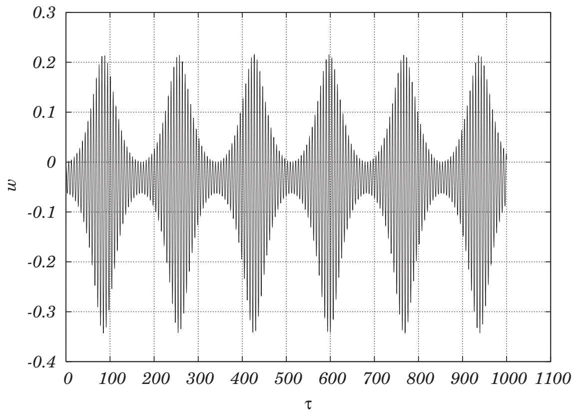

Figure 9. Solution of equation (23) for $g_1 = 0.2$, $g_2 = 0.03$ and initial conditions $w(0) = 0$ and $\dot{w}(0) = 0$. Low frequency modulation with large amplitude.

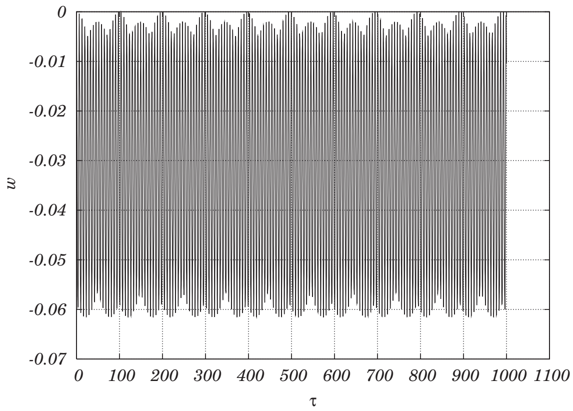

Figure 10. Solution of equation (23) for $g_1 = 0.1$, $g_2 = 0.03$ and $w(0) = 0$ and $\dot{w}(0) = 0$. High frequency modulation with small amplitude.

and the maximum is well above 0 (figure 9). Note that contrary to the field theory case (figure 1) there is no damping in the solution of simplified equation.

Solutions of the second group oscillate between some minimum value and 0. The modulation frequency can be both low and high (figure 10).

The third group of these solutions are solutions which have singularity at a certain time τ_{cr} (figure 11).

For large modulus of w, we can neglect linear terms in w and focus on the equation

$$\ddot{w} + w^2 = 0. \tag{24}$$

The behaviour of the solution for very large $|w|$ is given by

$$w \sim -6(\tau - \tau_{cr})^{-2}.$$

Of course, one cannot extrapolate these solution exactly to our partial differential equation (1) for obvious reasons. First of all the amplitude in a field theory cannot grow unlimited, but in this case pairs of soliton and antisoliton are created. This process can also happen when the amplitude is large enough so that A_d crosses below $-\frac{3\pi}{2}$.

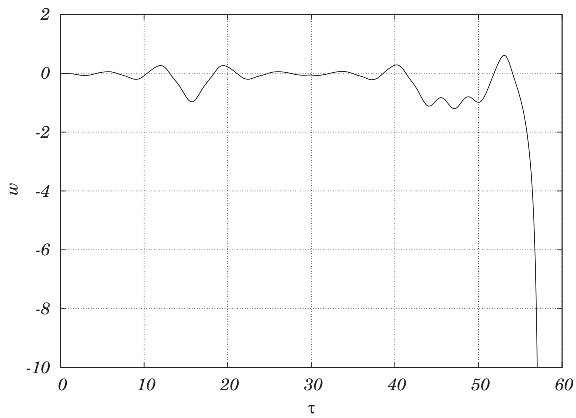

Figure 11. Solution of equation (23) for $g_1 = 0.9$, $g_2 = 0.03$ and $w(0) = 0$ and $\dot{w}(0) = 0$. Singularity around $t = 57$.

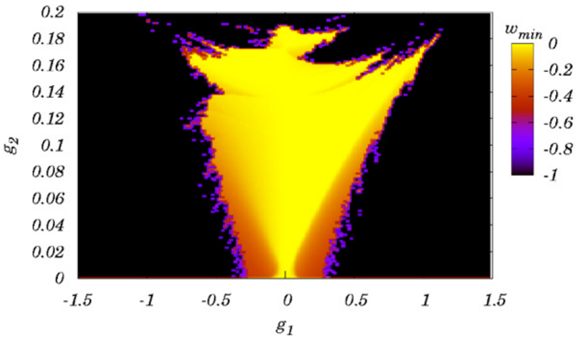

Figure 12. Stable and unstable (dark) solutions on a plane (g_1, g_2) with initial conditions $w(0) = 0$ and $\dot{w}(0) = 0$.

There is also one more interesting observation according to this simplified equation. The boundary between regular and singular solutions on a plane g_1, g_2 has a very complicated, presumably fractal, shape (figure 12). The same feature possesses the border between regular and singular solutions in a phase space of initial conditions with fixed g_1 and g_2 (figure 13). In fact, it is not surprising. Equation (12) possesses an invariance under discrete translation in time $t \to t + T$, where $T = \frac{2\pi}{\omega}$. If we knew the map in phase space $F : (A_d(t), \dot{A}_d(t)) \to (A_d(t+T), \dot{A}_d(t+T))$, we could just iterate F^n and find out whether for $n \to \infty$ the solutions are singular or not. The same procedure one usually applies in order to obtain a Julia's or Mandelbrot's sets, some of which are one of the most known fractals.

As a final result, we present figure 14 which shows the minima of the solutions of the simplified equation (12). Dark regions as usual represent the unstable solutions which certainly cross the level of $-\frac{3\pi}{2}$. For those conditions, we expect a pair of kink and antikink is produced.

After comparing these results with the results obtained for full partial differential equation (figure 5 and 7), we can see that there are discrepancies but the theoretical boundary lies not very far away from the true boundaries. We can see that again the picture reveals the fractal structure ($d_c = 1.60 \pm 0.03$, figure 15). The fractal dimension in this case is little smaller

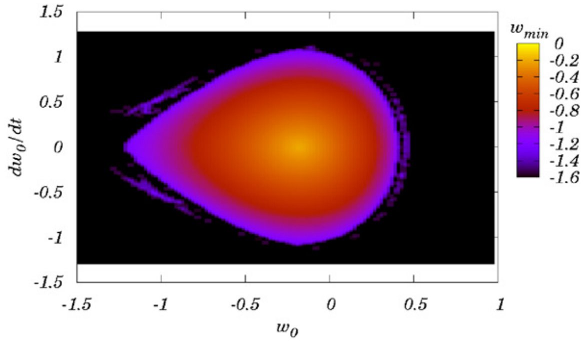

Figure 13. Stable and unstable solutions on a plane $(w_0, \dot{w}_0)$ for fixed $g_1 = 0.5$ and $g_2 = 0.12$.

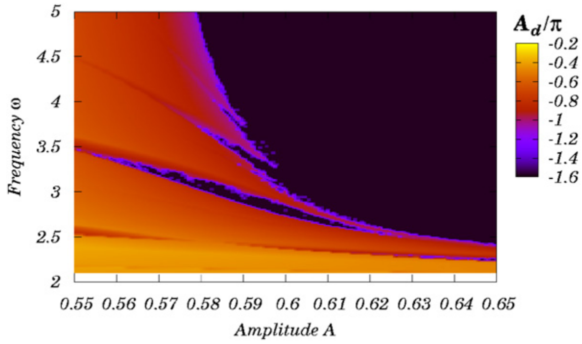

Figure 14. Minima of the solutions of the simplified equation in a plane of amplitude and frequency.

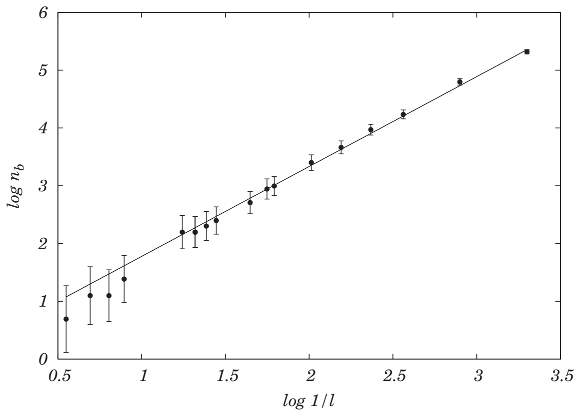

Figure 15. Dependence $\ln n_b$ upon $\ln 1/l$. The slope of the fitted line is 1.60 ± 0.03.

than in the two previous cases. Since we have taken only second-order simplified equation and neglected the escaping radiation, the 10% accuracy seems to be satisfying.

It is actually easy to explain if we consider the energy which is radiated from the oscillational mode. In the theoretical picture, we neglected this radiation. Of course, we also did not include the cubic term and we neglected the orthogonal modes, but even in that oversimplified picture it is clear that there is a similarity.

There is also one more interesting thing. Both in [8] and [11], the simplified theories which were based upon collective coordinates method reproduced the fractal structure which was wider than in field theory. In our case, we used the projection onto the internal mode of the soliton and our approach gave the structure which was much more narrow than in the partial differential equation.

4. Numerical methods

We have used the simplest possible discretization method for solving the full PDE (1):

$$\phi_j^i = \phi(t_j = j\lambda h, x_i = hi), \tag{25}$$

where $h = L/N$ is a size of the spatial grid spacing (L is the size of box, N is the number of grid points) and λh is the time step size and $\lambda < 1$, usually 0.1. The discretized spatial and time derivatives have the form

$$\frac{\partial^2}{\partial x^2}\phi_j^i = \frac{\phi_j^{i+1} + \phi_j^{i-1} - 2\phi_j^i}{h^2}, \tag{26}$$

$$\frac{\partial^2}{\partial t^2}\phi_j^i = \frac{\phi_{j+1}^i + \phi_{j-1}^i - 2\phi_j^i}{\lambda^2 h^2}, \tag{27}$$

and hence one can easily find the form for the field value in one time step:

$$\phi_{j+1}^i = 2\phi_j^i - \phi_{j-1}^i + \lambda^2\left(\phi_j^{i+1} + \phi_j^{i-1} - 2\phi_j^i\right) + \lambda^2 h^2 U'\left(\phi_j^i\right). \tag{28}$$

Our problem required only asymmetric solutions so we could limit only to the box $0 \leqslant x \leqslant L$. As we said in section 3 we had used two sets of initial and boundary conditions. The first one simulated the source switching far away from the kink:

$$\phi_0^i = \phi_1^i = \phi_s(\mathrm{i}h), \tag{29}$$

$$\phi_j^0 = 0, \tag{30}$$

$$\phi_j^N = \tfrac{1}{2}A\sin\omega\lambda hj. \tag{31}$$

In this particular problem, j must have been smaller than $3N/\lambda$ so that the waves could travel from the source (N/λ) go back to the boundary (again N/λ) and have no time to come back and interact with the kink again (less than N/λ).

The second set of conditions simulated the situation when the kink is put in standing wave:

$$\phi_0^i = \phi_s(\mathrm{i}h) + h_k(\mathrm{i}h), \tag{32}$$

$$\phi_1^i = \phi_s(\mathrm{i}h) + h_k(\mathrm{i}h)\cos\omega\lambda h, \tag{33}$$

$$\phi_j^0 = 0, \tag{34}$$

$$\phi_j^N = \phi_s(Nh) + h_k(Nh)\cos\omega\lambda jh. \tag{35}$$

In this case, we do not need to wait until the wave comes from the source so j must be less than $2N/\lambda$.

In order to find the best size h, which must compromise between reasonable calculation time and reliable precision, we have ran a few tests. By simply comparing the solutions for some h, $\frac{1}{2}h$ and $\frac{1}{4}h$, we found that even for $h = 0.05$ the discretizations h, $\frac{1}{2}h$ and $\frac{1}{4}h$ gave the same time for the first creation with accuracy $h\lambda$. Unfortunately, during the creation the system undergoes a very drastic change and the solutions after the creation differed. It did not matter to our problem, although it would be very interesting to find a similar fractal structure for further creations. After all we decided to use $h = 0.025$ and compare the results with $h = 0.0125$ and they all matched (for the first creation).

We also tested the method described above for linearized equation (5) where we expect that a travelling wave will not have any reflections. In fact, the numerical discretization gives reflection which is less than $20h^2$.

We simply used fourth-degree Runge–Kutta method with the time step 0.001 to solve the ODE (23) in the simplified model.

5. Conclusions and discussion

In the present paper, we have presented the evolution of a kink with an external perturbation (radiation) in ϕ^4 theory. We found that during the evolution an oscillating mode is being excited. We measured the excitation using numerical methods for two sets of initial and boundary conditions. Depending on the frequency of the radiation and its amplitude, the excitation of the oscillational mode can remain small enough to be treated only as a deformation of the kink or can grow and finally can effect in a production of kink and antikink which are radiated into the spatial infinity. We tried to explain this process using a simplified method by projecting our field theory equation onto the oscillational mode. The obtained equation is a nonlinear ordinary equation which we cannot solve analytically, but we may study it numerically or using perturbation methods. This equation possesses very rich space of solutions. Some of them are regular ones and some of them are singular. The border between those types of solutions is a very complicated both in a space of parameters and initial conditions. The solutions of the simplified equation are relatively good approximations of the solutions of the whole partial equation. The most interesting case of the creation of kink–antikink pair also reveals a quite well agreement between field theory and our simplified theory.

There are still many unanswered questions. First of them is why the border in the field theory is so complicated. We do not know how to add a damping term which we think is very important to solve the problem. Our simplified theory does not give the answers about the dynamics of the created defects. We do not know how much energy from the oscillational mode is converted into the kinetic energy of kinks, how much into radiation and how much energy remains in oscillating modes around all the defects.

Acknowledgments

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